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VISIBILITY OF CARTESIAN PRODUCTS OF CANTOR SETS

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Abstract

Let K_λ be the attractor of the following iterated function system(IFS):

$$\{f_1(x) = \lambda x,\ f_2(x) = \lambda x + 1 - \lambda\}, \quad 0 < \lambda < 1/2.$$

Given $\alpha \geq 0$, we say the line $y = \alpha x$ is visible through $K_\lambda \times K_\lambda$ if

$$\{(x, \alpha x) : x \in \mathbb{R}\backslash\{0\}\} \cap (K_\lambda \times K_\lambda) = \emptyset.$$

Let $V = \{\alpha \geq 0 : y = \alpha x$ is visible through $K_\lambda \times K_\lambda\}$. In this paper, we give a complete description of V, containing its Hausdorff dimension and topological properties.

Keywords: Visible Part; Cantor Sets; Hausdorff Dimension.

1. INTRODUCTION

Projections, sections, geodesic curves and visibility are related to many aspects of fractal geometry. For instance, the arithmetic sum of two self-similar sets is indeed the projection problem[1,2]; sections of some fractal sets are connected to the multiple representations of real numbers[3]; geodesic curves on fractal sets are distinct from the classical differential

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manifolds.[4] For more results on these problems, we refer to Refs. 5–8.

The concept of "visibility" was investigated by many scholars. Nikodym constructed a subset F of $\mathbb{R}^2$ such that every point of F is visible from two diametrically opposite directions.[9] In convex geometry, Krasnosel offered a beautiful criterion which enables us to check whether the entire boundary of a compact set of $\mathbb{R}^2$ is visible from an interior point.[10] Falconer and Fraser proved that for a class of plane self-similar sets when the attractor F has Hausdorff dimension greater than 1 the Hausdorff dimension of the visible subset is 1.[11] The visible problem is also related to the arithmetic on the fractals.[12] The readers can find more related results in Refs. 13–16. Given $\alpha \geq 0$ and some subset $F \subset \mathbb{R}^2$, we say the line $y = \alpha x$ is visible through F if

$$\{(x, \alpha x) : x \in \mathbb{R}\backslash\{0\}\} \cap F = \emptyset.$$

The middle-third Cantor set is one of the oldest mathematical examples of fractals, which was constructed by George Cantor in 1883.[17] As the best example of a perfect nowhere-dense set in the real line, the middle-third Cantor set has already been studied by many scholars, see Refs. 18, 19 for instance and references therein. There are also some results on the variations and higher-dimensional extending of the classical middle-third Cantor set. Zou *et al*. considered the self-similar structure on the intersection of the middle-$(1-2\lambda)$ Cantor set with $\lambda \in (1/3, 1/2)$.[20] Deng *et al*. characterized the shape of an optimal set of Cartesian product of the middle-third Cantor set.[21] In this paper, we shall consider the visibility of the Cartesian products of the middle-$(1-2\lambda)$ Cantor set with $\lambda \in (0, 1/2)$.

Let us first recall that a map $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ is termed a contracting similarity if there exists a number r with $0 < r < 1$ such that $|\varphi(x) - \varphi(y)| = r|x - y|$ for all $x, y \in \mathbb{R}^N$. Let $\{\varphi_i\}_{i=1}^m$ be an iterated function system (IFS) consisting of finite contracting similarities. It is well known that there exists a unique nonempty compact set F such that $F = \bigcup_{i=1}^m \varphi_i(F)$. We call F the self-similar set or attractor for the IFS $\{\varphi_i\}_{i=1}^m$.[18] We say that the IFS satisfies the open set condition if there exists a nonempty open set O such that $\bigcup_{i=1}^m \varphi_i(O) \subseteq O$ with the union disjoint. In this paper, we shall analyze the following Cantor set.

Let K_λ be the attractor for the IFS

$$\{f_1(x) = \lambda x, f_2(x) = \lambda x + 1 - \lambda\}, \quad 0 < \lambda < 1/2,$$

i.e.

$$K_\lambda = f_1(K_\lambda) \cup f_2(K_\lambda). \tag{1}$$

Observe that K_λ can be obtained from the closed interval $[0, 1]$ by removing a fixed proportion of each subinterval in each of the iterative steps. In other words, K_λ is a middle-$(1-2\lambda)$ Cantor set.[22]

Let

$$V = \{\alpha \geq 0 : y = \alpha x \text{ is visible through } K_\lambda \times K_\lambda\}.$$

It is easy to verify that the line $y = \alpha x$ is visible through $K_\lambda \times K_\lambda$ if and only if

$$\alpha \notin \frac{K_\lambda}{K_\lambda\backslash\{0\}} := \left\{\frac{x}{y} : x, y \in K_\lambda, y \neq 0\right\}.$$

Thus,

$$V = [0, +\infty) \Big\backslash \frac{K_\lambda}{K_\lambda\backslash\{0\}}. \tag{2}$$

By $A^o, m(A)$ and $\dim_H(A)$ we denote the set of interior points, the Lebesgue measure and the Hausdorff dimension of A, respectively. Now, we state our results.

Theorem 1.1. *Let K_λ be given by* (1). *Then*

(1) *When* $\frac{3-\sqrt{5}}{2} \leq \lambda < \frac{1}{2}$, $V = \emptyset$.

(2) *When* $\frac{1}{3} \leq \lambda < \frac{3-\sqrt{5}}{2}$,
$$V = (0, +\infty) \Big\backslash \bigcup_{k=-\infty}^{\infty} \lambda^k \left[1 - \lambda, \frac{1}{1-\lambda}\right].$$

(3) *When* $0 < \lambda < \frac{1}{3}$, $V^o \neq \emptyset$. *In particular, when* $\frac{1}{4} < \lambda < \frac{1}{3}$, $([0, +\infty)\backslash V)^o \neq \emptyset$; *when* $0 < \lambda \leq \frac{1}{4}$, $m([0, +\infty)\backslash V) = 0$ *and* $\dim_H([0, +\infty)\backslash V) = \frac{\log 4}{-\log \lambda}$.

The paper is arranged as follows. In Sec. 2, we give the proof of Theorem 1.1. Finally, we give some remarks.

2. PROOFS OF MAIN RESULTS

First, we introduce some notations. Let $E = [0, 1]$. For any $(i_1, \ldots, i_n) \in \{1, 2\}^n$, we call $f_{i_1,\ldots,i_n}([0, 1]) = (f_{i_1} \circ \cdots \circ f_{i_n})([0, 1])$ a basic interval of rank n, which has length λ^n. Denote by E_n the collection of all these basic intervals of rank n. Suppose A and B are the left and right endpoints of some basic intervals in E_k for some $k \geq 1$, respectively. Denote by G_n the union of all the basic intervals of rank n which are contained in $[A, B]$. Let I be a basic interval with rank n. Define $\widetilde{I} = f_1(I) \cup f_2(I)$.

The following lemma comes from Refs. 23 and 24, here we give its proof just for the integrity of the content.

Lemma 2.1. *Let $F : U \to \mathbb{R}$ be a continuous function, where $U \subset \mathbb{R}^2$ is a nonempty open set. Suppose A and B are the left and right endpoints of some basic intervals in G_{k_0} for some $k_0 \geq 1$, respectively, such that $[A, B] \times [A, B] \subset U$. Then $K_\lambda \cap [A, B] = \bigcap_{n=k_0}^{\infty} G_n$. Moreover, if for any $n \geq k_0$ and any two basic intervals $I, J \subset G_n$,*

$$F(I, J) = F(\widetilde{I}, \widetilde{J}),$$

where $F(I, J) := \{F(x, y) : x \in I, y \in J\}$, then

$$F(K_\lambda \cap [A, B], K_\lambda \cap [A, B]) = F(G_{k_0}, G_{k_0}).$$

Proof. By the construction of G_n, i.e. $G_{n+1} \subset G_n$ for any $n \geq k_0$, it follows that

$$K_\lambda \cap [A, B] = \bigcap_{n=k_0}^{\infty} G_n.$$

The continuity of F yields that

$$F(K_\lambda \cap [A, B], K_\lambda \cap [A, B]) = \bigcap_{n=k_0}^{\infty} F(G_n, G_n).$$

Without loss of generality, we may assume that

$$G_n = \cup_{1 \leq i \leq t_n} I_{n,i} \quad \text{for some } t_n \geq 1,$$

where $I_{n,i}$ is a basic interval in G_n. By the condition in lemma, i.e. for any $n \geq k_0$ and any two basic intervals $I, J \subset G_n$, such that

$$F(I, J) = F(\widetilde{I}, \widetilde{J}),$$

it follows that

$$\begin{aligned} F(G_n, G_n) &= \cup_{1 \leq i \leq t_n} \cup_{1 \leq j \leq t_n} F(I_{n,i}, I_{n,j}) \\ &= \cup_{1 \leq i \leq t_n} \cup_{1 \leq j \leq t_n} F(\widetilde{I_{n,i}}, \widetilde{I_{n,j}}) \\ &= F(\cup_{1 \leq i \leq t_n} \widetilde{I_{n,i}}, \cup_{1 \leq j \leq t_n} \widetilde{I_{n,j}}) \\ &= F(G_{n+1}, G_{n+1}). \end{aligned}$$

Therefore, $F(K_\lambda \cap [A, B], K_\lambda \cap [A, B]) = F(G_{k_0}, G_{k_0})$. □

Lemma 2.2. *Let $f(x, y) = \frac{x}{y}$, and $I = [a, a + t], J = [b, b + t]$ be two basic intervals. If $1/3 \leq \lambda < 1/2$, and $b \geq a \geq 1 - \lambda$, then $f(\tilde{I}, \tilde{J}) = f(I, J)$.*

Proof. Note that

$$\tilde{I} = [a, a + \lambda t] \cup [a + t - \lambda t, a + t],$$

$$\tilde{J} = [b, b + \lambda t] \cup [b + t - \lambda t, b + t].$$

Therefore,

$$f(\tilde{I}, \tilde{J}) = J_1 \cup J_2 \cup J_3 \cup J_4,$$

where

$$\begin{aligned} J_1 &= \left[\frac{a}{b+t}, \frac{a+\lambda t}{b+t-\lambda t}\right] =: [r_1, s_1], \\ J_2 &= \left[\frac{a}{b+\lambda t}, \frac{a+\lambda t}{b}\right] =: [r_2, s_2], \\ J_3 &= \left[\frac{a+t-\lambda t}{b+t}, \frac{a+t}{b+t-\lambda t}\right] =: [r_3, s_3], \\ J_4 &= \left[\frac{a+t-\lambda t}{b+\lambda t}, \frac{a+t}{b}\right] =: [r_4, s_4]. \end{aligned}$$

Note that $f(I, J) = [r_1, s_4]$. In the following, we verify that $f(I, J) = J_1 \cup J_2 \cup J_3 \cup J_4$.

Since $b \geq a \geq 1 - \lambda$ and $\lambda \geq \frac{1}{3}$, we have

$$\begin{aligned} r_3 - r_2 &= \frac{a+t-\lambda t}{b+t} - \frac{a}{b+\lambda t} \\ &= \frac{t(1-\lambda)(b-a+t\lambda)}{(b+t\lambda)(b+t)} \geq 0. \end{aligned}$$

Now, it suffices to check that

$$s_1 - r_2 \geq 0, \quad s_2 - r_3 \geq 0 \quad \text{and} \quad s_3 - r_4 \geq 0.$$

We have

$$\begin{aligned} s_1 - r_2 &= \frac{a+\lambda t}{b+t-\lambda t} - \frac{a}{b+\lambda t} \\ &= \frac{t(2a\lambda - a + b\lambda + t\lambda^2)}{(b+t-\lambda t)(b+\lambda t)} \\ &\geq \frac{t(a(3\lambda-1) + t\lambda^2)}{(b+t-\lambda t)(b+\lambda t)} \geq 0, \end{aligned}$$

and

$$\begin{aligned} s_2 - r_3 &= \frac{a+\lambda t}{b} - \frac{a+t-\lambda t}{b+t} \\ &= \frac{t(a+(2\lambda-1)b+t\lambda)}{b(b+t)} \\ &\geq \frac{t(1-\lambda+(2\lambda-1)b+t\lambda)}{b(b+t)} \geq 0. \end{aligned}$$

Finally,

$$\begin{aligned} s_3 - r_4 &= \frac{a+t}{b+t-\lambda t} - \frac{a+t-\lambda t}{b+\lambda t} \\ &= \frac{t(-a-t+2a\lambda+b\lambda+3t\lambda-t\lambda^2)}{(b+t-\lambda t)(b+\lambda t)}. \end{aligned}$$

If $b \neq a$, then $b > a + t$. Therefore, we have

$$\begin{aligned} &-a - t + 2a\lambda + b\lambda + 3t\lambda - t\lambda^2 \\ &\quad \geq -a + 2a\lambda + (a+t)\lambda + t(3\lambda - 1 - \lambda^2) \\ &\quad \geq a(3\lambda - 1) + t(4\lambda - 1 - \lambda^2) \geq 0, \end{aligned}$$

which leads to $s_3 - r_4 \geq 0$. However, if $a = b$, then

$$s_2 - r_4 = \frac{a+\lambda t}{a} - \frac{a+t-\lambda t}{a+\lambda t} = \frac{\lambda^2 t^2 + at(3\lambda - 1)}{a(a+\lambda t)} \geq 0.$$

Thus, we finish checking that $f(I, J) = J_1 \cup J_2 \cup J_3 \cup J_4 = [r_1, s_4] = \left[\frac{a}{b+t}, \frac{a+t}{b}\right]$. □

Lemma 2.3. *We have*

$$\frac{K_\lambda}{K_\lambda \backslash \{0\}} = \begin{cases} [0, \infty), & \text{when } \frac{3-\sqrt{5}}{2} \leq \lambda < \frac{1}{2}, \\ \bigcup_{k=-\infty}^{+\infty} \lambda^k \left[1-\lambda, \frac{1}{1-\lambda}\right] \cup \{0\}, & \text{when } \frac{1}{3} \leq \lambda < \frac{3-\sqrt{5}}{2}. \end{cases}$$

Proof. From Lemmas 2.1 and 2.2, it follows that if $\lambda \geq \frac{1}{3}$

$$\frac{f_2(K_\lambda)}{f_2(K_\lambda)} = \left[1-\lambda, \frac{1}{1-\lambda}\right].$$

Each $x \in K_\lambda$ can be uniquely represented as

$$x = \sum_{n=1}^{\infty} x_n \lambda^n \quad \text{with } x_n \in \{0, 1-\lambda\}.$$

Note that $x \in f_2(K_\lambda)$ if and only if $x_1 = 1 - \lambda$. Thus, each $x \in K_\lambda \backslash \{0\}$ is of form

$$x = \lambda^m x^* \quad \text{with } m \in \{0, 1, 2, \ldots\}, \quad x^* \in f_2(K_\lambda).$$

Thus, for any two $x = \lambda^m x^*, y = \lambda^n y^* \in K_\lambda \backslash \{0\}$ with $x^*, y^* \in f_2(K_\lambda)$ one has

$$\frac{x}{y} = \lambda^{m-n} \cdot \frac{x^*}{y^*} \in \lambda^{m-n}\left[1-\lambda, \frac{1}{1-\lambda}\right].$$

Thus,

$$\frac{K_\lambda}{K_\lambda \backslash \{0\}} = \{0\} \cup \bigcup_{k=-\infty}^{\infty} \lambda^k \left[1-\lambda, \frac{1}{1-\lambda}\right].$$

It is easy to check that $\bigcup_{k=-\infty}^{\infty} \lambda^k [1-\lambda, \frac{1}{1-\lambda}] \cup \{0\} = [0, +\infty)$ when $\frac{3-\sqrt{5}}{2} \leq \lambda < \frac{1}{2}$, and intervals $\lambda^k [1-\lambda, \frac{1}{1-\lambda}]$ are pairwise disjoint when $\frac{1}{3} \leq \lambda < \frac{3-\sqrt{5}}{2}$. □

For K_{λ_1} and K_{λ_2} with $\frac{\log \lambda_1}{\log \lambda_2} \in \mathbb{Q}$, i.e. there exist $m_0, n_0 \in \mathbb{Z}$ such that $\frac{\log \lambda_1}{\log \lambda_2} = \frac{n_0}{m_0}$ and $(m_0, n_0) = 1$, Pourbarat proved the following result (Ref. 25, Corollary 10).

Theorem 2.4. *If* $\frac{\lambda_1}{1-2\lambda_1} \cdot \frac{\lambda_2}{1-2\lambda_2} > \frac{1}{\gamma}$, *then* $\frac{K_{\lambda_1}}{K_{\lambda_2} \backslash \{0\}}$ *contains an interior point, where* $\gamma := \lambda_1^{-\frac{1}{n_0}}$.

Lemma 2.5. *If* $\lambda > \frac{1}{4}$, *then* $\frac{K_\lambda}{K_\lambda \backslash \{0\}}$ *contains an interior point.*

Proof. If $\lambda > \frac{1}{4}$, then $\frac{\lambda^2}{(1-2\lambda)^2} > \lambda$. Therefore, $\frac{K_\lambda}{K_\lambda \backslash \{0\}}$ contains an interior point by Theorem 2.4. □

Lemma 2.6. *If* $0 < \lambda < \frac{3-\sqrt{5}}{2}$, *then* V *has an interior point.*

Proof. Note that $f_2(K_\lambda) \subset [1-\lambda, 1]$. Thus, by the argument in Lemma 2.3, we have

$$\frac{K_\lambda}{K_\lambda \backslash \{0\}} \subseteq \{0\} \cup \bigcup_{k=-\infty}^{\infty} \lambda^k \left[1-\lambda, \frac{1}{1-\lambda}\right].$$

Note that the intervals $\left[\lambda^k(1-\lambda), \frac{\lambda^k}{1-\lambda}\right]$ for $k \in \mathbb{Z}$ are pairwise disjoint when $0 < \lambda < \frac{3-\sqrt{5}}{2}$. Therefore, V has an interior point by (2). □

We recall that $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. A Borel set A is a 1-set if $0 < \mathcal{H}^1(A) < \infty$. Simon and Solomyak showed that if Λ is a self-similar 1-set in the plane satisfying the open set condition and not contained in a line, then it has radial projection of zero length from every point.[26] Fixing the point at the origin, we obtain the following result.

Theorem 2.7. *Let* Λ *be a self-similar* 1*-set in* $\mathbb{R}^2$ *with the open set condition, which is not on a line. Then*

$$m(P_{(0,0)}(\Lambda \backslash \{(0,0)\})) = 0,$$

where

$$P_{(0,0)} : \mathbb{R}^2 \backslash (0,0) \to S^1, P_{(0,0)}(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|}.$$

Lemma 2.8. $\frac{K_{1/4}}{K_{1/4} \backslash \{0\}}$ *has Lebesgue measure zero.*

Proof. Note that when $\lambda = 1/4$, $\Lambda = K_\lambda \times K_\lambda$ is a self-similar set with the following IFS

$$g_1(x, y) = \left(\frac{x}{4}, \frac{y}{4}\right),$$

$$g_2(x, y) = \left(\frac{x+3}{4}, \frac{y}{4}\right),$$

$$g_3(x,y) = \left(\frac{x+3}{4}, \frac{y+3}{4}\right),$$

$$g_4(x,y) = \left(\frac{x}{4}, \frac{y+3}{4}\right).$$

Clearly, the above IFS satisfies the open set condition. Therefore, the Hausdorff dimension of Λ is 1, and $0 < \mathcal{H}^1(\Lambda) < \infty$. Let

$$\Gamma = \left\{\frac{(x,y)}{\sqrt{x^2+y^2}} \in S^1 : (x,y) \in K_\lambda \times K_\lambda \backslash \{(0,0)\}\right\}$$
$$= P_{(0,0)}(\Lambda \backslash \{(0,0)\}).$$

The Lebesgue measure of Γ is 0 due to Theorem 2.7. Let

$$\Gamma_1 = \left\{\frac{(x,y)}{\sqrt{x^2+y^2}} \in S^1 : (x,y) \in K_\lambda \times K_\lambda \backslash \{(0,0)\}, x \neq 0\right\}.$$

Clearly, $m(\Gamma_1) = m(\Gamma) = 0$. The metric on Γ_1, denoted by d_1, is the arc metric. It is well known that on S^1, the arc metric is equivalent to the Euclidean metric. Let

$$\Gamma_2 = \left\{\arctan\frac{y}{x} : (x,y) \in K_\lambda \times K_\lambda \backslash \{(0,0)\}, x \neq 0\right\}.$$

The metric on Γ_2 is the Euclidean metric (we denote it by d_2). We define the the map

$$\phi : \Gamma_1 \to \Gamma_2,$$

by

$$\phi\left(\frac{(x,y)}{\sqrt{x^2+y^2}}\right) = \arctan\frac{y}{x}.$$

The map ϕ is indeed mapping a point on S^1 into its associated polar angle in the polar coordinate system. Therefore, we may define ϕ in another way as follows:

$$\phi : \Gamma_1 \to \Gamma_2, \quad \phi(\mathbf{a}) = \theta_{\mathbf{a}}.$$

Clearly, ϕ is well-defined, and it is a bijection. Moreover, we shall prove that ϕ is a Lipschitz map, i.e. there exists some constant $L > 0$ such that

$$d_2(\phi(\mathbf{a}), \phi(\mathbf{b})) \leq L d_1(\mathbf{a}, \mathbf{b}).$$

Note that $d_2(\phi(\mathbf{a}), \phi(\mathbf{b})) = d_2(\theta_{\mathbf{a}}, \theta_{\mathbf{b}})$, and that

$$d_1(\mathbf{a}, \mathbf{b}) = d_2(\theta_{\mathbf{a}} \cdot 1, \theta_{\mathbf{b}} \cdot 1) = d_2(\theta_{\mathbf{a}}, \theta_{\mathbf{b}}).$$

Now, $m(\Gamma_2) = 0$ follows from $\phi(\Gamma_1) = \Gamma_2$, $m(\Gamma_1) = 0$, and ϕ is Lipschitz. Therefore, $m\left(\frac{K_{1/4}}{K_{1/4}\backslash\{0\}}\right) = 0$. □

The following is proved in Ref. [27, Theorem 2.7].

Theorem 2.9. *Let Λ be an arbitrary self-similar set in $\mathbb{R}^2$ not contained in any line. Suppose that $g : \mathbb{R}^2 \to \mathbb{R}$ is a C^2 map such that*

$$(g_x)^2 + (g_y)^2 \neq 0,$$
$$(g_{xx}g_y - g_{xy}g_x)^2 + (g_{xy}g_y - g_{yy}g_x)^2 \neq 0$$

for any $(x,y) \in \Lambda$. Then

$$\dim_H g(\Lambda) = \min\{1, \dim_H(\Lambda)\}.$$

Lemma 2.10. *When $0 < \lambda \leq \frac{1}{4}$, $\dim_H([0, +\infty)\backslash V) = \frac{\log 4}{-\log\lambda}$.*

Proof. By the argument in Lemma 2.3, we have

$$\frac{K_\lambda}{K_\lambda\backslash\{0\}} = \bigcup_{k=-\infty}^{\infty} \lambda^k \frac{f_2(K_\lambda)}{f_2(K_\lambda)} \cup \{0\}.$$

Thus

$$\dim_H \frac{K_\lambda}{K_\lambda\backslash\{0\}} = \dim_H \frac{f_2(K_\lambda)}{f_2(K_\lambda)}.$$

Clearly, $\Lambda = f_2(K_\lambda) \times f_2(K_\lambda)$ is a two-dimensional self-similar set which is not contained in any line. Let $g(x,y) = \frac{x}{y}$, then

$$(g_x)^2 + (g_y)^2 \neq 0,$$
$$(g_{xx}g_y - g_{xy}g_x)^2 + (g_{xy}g_y - g_{yy}g_x)^2 \neq 0$$

for any $(x,y) \in \Lambda$. Therefore, in terms of Theorem 2.9,

$$\dim g(\Lambda) = \dim_H \frac{f_2(K_\lambda)}{f_2(K_\lambda)}$$
$$= \min\{\dim_H(f_2(K_\lambda) \times f_2(K_\lambda)), 1\}$$
$$= \min\{2\dim_H(K_\lambda), 1\}.$$

Hence, if $0 < \lambda \leq 1/4$, then

$$\dim_H \frac{f_2(K_\lambda)}{f_2(K_\lambda)} = 2\dim_H(K) = \frac{\log 4}{-\log\lambda}.$$ □

Proof of Theorem 1.1. Theorem 1.1(1) and (2) follows from Lemmas 2.3. Theorem 1.1(3) follows from Lemmas 2.5, 2.6, 2.8 and 2.10. □

3. FINAL REMARKS

The main idea of this paper is to establish a connection between the visible problem and arithmetic on the fractal sets. Our idea can be implemented for other overlapping self-similar sets. Similar results can be obtained if we replace the line $y = \alpha x$ in the definition of V, by some parabolic curves or hyperbolic curves. However, these cases are more complicate. We shall discuss these problems in another paper.

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